CALCULATION OF THE VARIATION OF SMALL AMPLITUDE PLANE WAVE SPEEDS WITH PRESSURE

Uniform compression is the simplest of initially deformed configurations from an experimental as well as theoretical point of view, permitting elastic response of real crystals for deformations which may be large.

Setting

$$\xi_{\mathrm{k}} = G_{\mathrm{k}\alpha} X_\alpha = \lambda \delta_{\mathrm{k}\alpha} X_\alpha, \quad 0 < \lambda < 1 \tag{24}$$

then

$$C_{\alpha\beta} = \xi_{\mathrm{k},\alpha} \xi_{\mathrm{l},\beta} \delta_{\mathrm{kl}} = \lambda^2 \delta_{\alpha\beta}\,. \tag{25}$$

In matrix representation

$$C = (C_{\alpha\beta}) = \lambda^2 I\,,$$
$$C^{-1} = (C^{-1}_{\alpha\beta}) = 1/\lambda^2 I\,. \tag{26}$$

Relative to the homogeneously compressed configuration $\tilde{K}$

$$4\frac{\partial^2 \Sigma}{\partial \tilde{C}_{\kappa\lambda} \partial \tilde{C}_{\mu\nu}} = \left[4\frac{\partial^2 \Sigma}{\partial C_{\kappa\lambda} \partial C_{\mu\nu}}\right]_{C=\lambda^2 I} = \tilde{C}_{\kappa\lambda\mu\nu}(\lambda) \tag{27}$$

$$2\frac{\partial \Sigma}{\partial \tilde{C}_{\kappa\lambda}} = \left[2\frac{\partial \Sigma}{\partial C_{\kappa\lambda}}\right]_{C=\lambda^2 I} = \tilde{T}_{\kappa\lambda}(\lambda) = -\lambda \tilde{p}(\lambda)\delta_{\kappa\lambda} \tag{28}$$

so that

$$\tilde{Q}_{\mathrm{kl}}(\boldsymbol{v}) = [\lambda^4 \tilde{C}_{\kappa\lambda\mu\nu} \delta_{\mathrm{p}\kappa} \delta_{\mathrm{k}\lambda} \delta_{\mathrm{q}\mu} \delta_{\mathrm{l}\nu} - \lambda^3 \tilde{p} \delta_{\mathrm{pq}} \delta_{\mathrm{kl}}] v_{\mathrm{p}} v_{\mathrm{q}}\,, \tag{29}$$

where $\tilde{C}_{\kappa\lambda\mu\nu}(\lambda)$ are the second-order elastic constants in the compressed state and $\tilde{p}(\lambda)$ the initial pressure. The $\tilde{Q}_{\mathrm{kl}}(\boldsymbol{v})$ can be theoretically calculated for any propagation direction $\boldsymbol{v}$ by use of eqns. (1) and (26) to calculate $\tilde{C}_{\kappa\lambda\mu\nu}(\lambda)$ and $\tilde{p}(\lambda)$[2].

The square of the wave speeds follows from eqns. (22) and (29) as

$$U^2(\boldsymbol{v}) = \frac{1}{\rho_0} \frac{A_{\mathrm{k}} A_{\mathrm{l}}}{|A|^2} \tilde{Q}_{\mathrm{kl}}(\boldsymbol{v}) = \frac{1}{\rho_0} \frac{A_{\mathrm{k}} A_{\mathrm{l}}}{|A|^2} \{\lambda^4 \tilde{C}_{\kappa\lambda\mu\nu} \delta_{\mathrm{p}\kappa} \delta_{\mathrm{k}\lambda} \delta_{\mathrm{q}\mu} \delta_{\mathrm{l}\nu} - \lambda^3 \tilde{p} \delta_{\mathrm{pq}} \delta_{\mathrm{kl}}\} v_{\mathrm{p}} v_{\mathrm{q}} \tag{30}$$

with real values when the right side is positive. In the compressed configuration $\tilde{K}$ the second-order elastic constants have the same symmetry properties (body-centered cubic) as in the natural state:

$$\tilde{C}_{1111} = \tilde{C}_{2222} = \tilde{C}_{3333}$$
$$\tilde{C}_{2323} = \tilde{C}_{1313} = \tilde{C}_{1212}$$
$$\tilde{C}_{1122} = \tilde{C}_{1133} = \tilde{C}_{2233}\,, \tag{31}$$

all other $\tilde{C}_{\kappa\lambda\mu\nu}$ vanishing. Furthermore, as has been shown both experimentally[15] and theoretically[2] for sodium

$$0 < \tilde{C}_{2323} < \tilde{C}_{1122} < \tilde{C}_{1111}$$
$$0 < \tilde{C}_{1111} - \tilde{C}_{1122} < \tilde{C}_{2323} \qquad 0 < \lambda \leqslant 1$$
$$0 < \tilde{C}_{1111} - \tilde{C}_{1122} < \tilde{C}_{1111} + \tilde{C}_{1122}\,. \tag{32}$$

These inequalities apply for all alkali metals in the natural state, *i.e.*, $\lambda = 1$[2], and can be expected to hold when $\lambda < 1$ for lithium, potassium, rubidium and cesium as well.

For a plane wave front propagating in the ξ_1 direction, $\boldsymbol{v}$:(1, 0, 0), the hyperelastic b.c.c. crystal

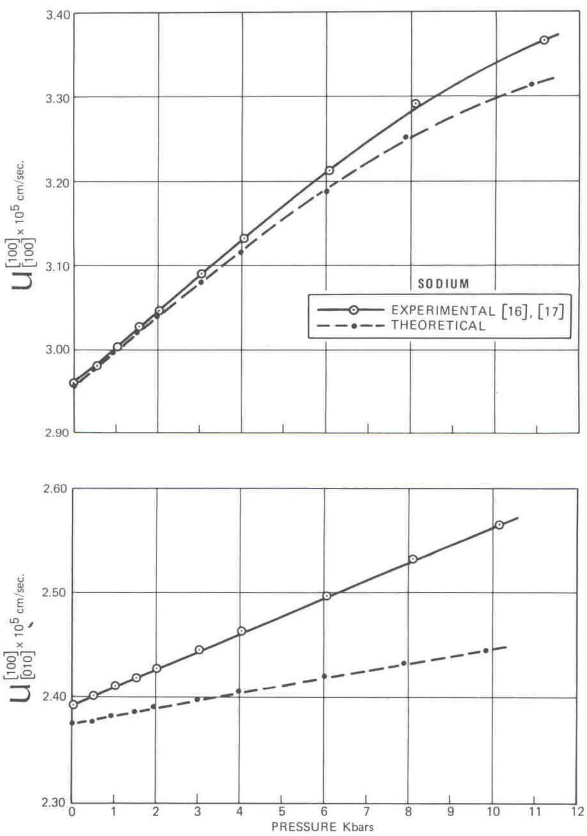

Fig. 1. Variation of $U^{[100]}_{[100]}$ and $U^{[100]}_{[010]}$ with pressure for sodium.

admits pure longitudinal and transverse waves with propagation speeds

$$U^{[100]}_{[100]} = U_{\mathrm{L}} = \left\{ \frac{1}{\rho_0} \tilde{Q}_{11}[100] \right\}^{\frac{1}{2}} = \left\{ \frac{\lambda^3}{\rho_0} [\lambda \tilde{C}_{1111} - \tilde{p}] \right\}^{\frac{1}{2}} \tag{33}$$

$$U^{[100]}_{[010]} = U^{[100]}_{[001]} = U_{\mathrm{T}} = \left\{ \frac{1}{\rho_0} \tilde{Q}_{22}[100] \right\}^{\frac{1}{2}} = \left\{ \frac{\lambda^3}{\rho_0} [\lambda \tilde{C}_{2323} - \tilde{p}] \right\}^{\frac{1}{2}} \tag{34}$$

where (32) requires

$$U_{\mathrm{T}} < U_{\mathrm{L}} \,. \tag{35}$$

The upper brace of indices designates the wave front direction while the lower brace denotes the displacement direction or acoustic axes. Calculated values of the wave speeds for sodium and potassium together with experimental data are shown in Figs. 1 and 2. The experimental curves are constructed from the low temperature elastic coefficient data[16,17] and the room temperature data on the pressure variation of the coefficients[18,19]. The theoretical curves are of course for absolute zero temperature. The rate of increase of wave speed with pressure is seen to be greater for the longitudinal waves, both theoretically and experimentally. It is of interest therefore to note that in addition to inequality (35), both theoretical and experimental results further satisfy Truesdell's stronger inequality[14]

$$\tfrac{4}{3} U_{\mathrm{T}}^2 < U_{\mathrm{L}}^2 \tag{36}$$

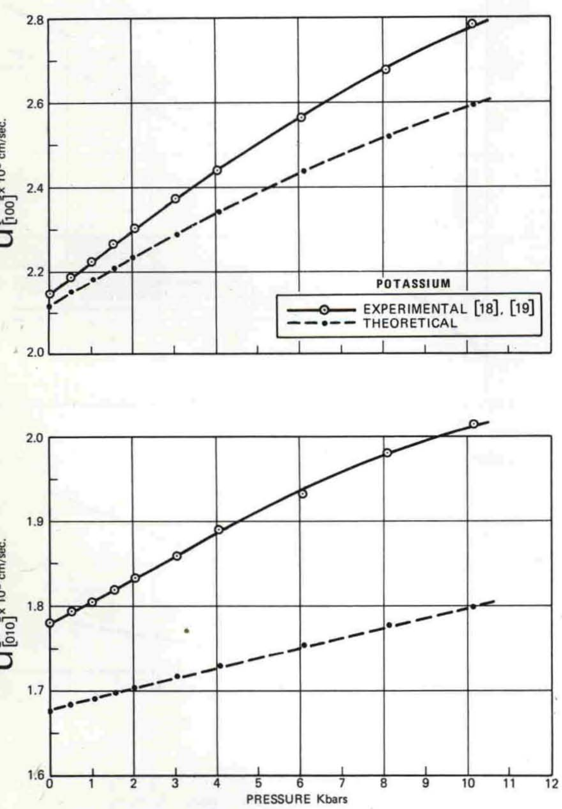

Fig. 2. Variation of $U^{[100]}_{[100]}$ and $U^{[100]}_{[010]}$ with pressure for potassium.

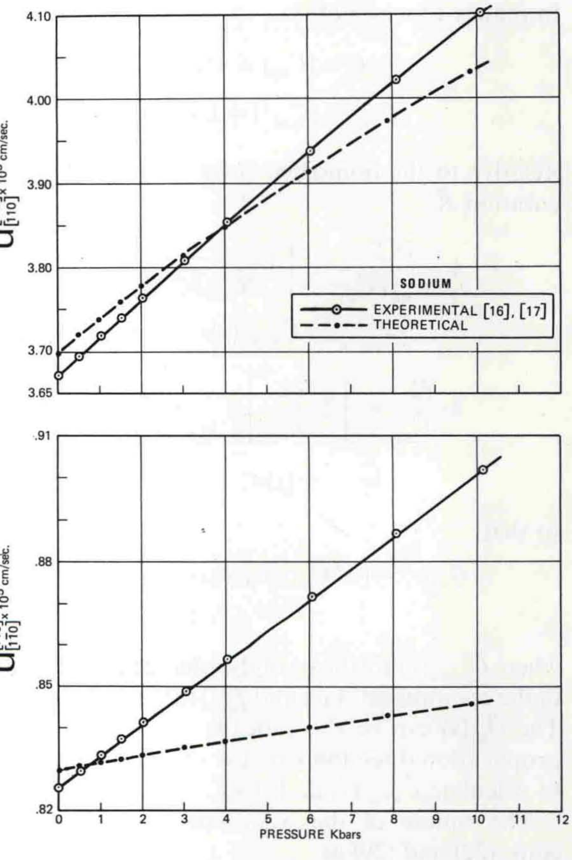

Fig. 3. Variation of $U^{[110]}_{[110]}$ and $U^{[110]}_{[1\bar{1}0]}$ with pressure for sodium.

which is a "universal relation", that is, which holds for any state of initial pressure in every solid deforming elastically.

For a plane wave front traveling in the [110] direction, $\boldsymbol{v}: (1/\sqrt{2}, 1/\sqrt{2}, 0)$, there is associated an orthogonal triad of waves with speeds